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THE HIGHER DERIVATIVE EXPANSION OF THE EFFECTIVE ACTION BY THE STRING–INSPIRED METHOD, PART I

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ABSTRACT

The higher derivative expansion of the one-loop effective action for an external scalar potential is calculated to order $O(T^7)$, using the string-inspired Bern-Kosower method in the first quantized path integral formulation. Comparisons are made with standard heat kernel calculations and with the corresponding Feynman diagrammatic calculation in order to show the efficiency of the present method.

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As Bern and Kosower have shown [1], the possibility of representing amplitudes in ordinary quantum field theory as the infinite string tension limit of certain superstring amplitudes allows the derivation of a new set of Feynman rules for one-loop calculations which, though very different from the usual ones, appear to be completely equivalent [2]. These rules lead to a significant reduction of the number of terms to be computed in gauge theory calculations, as they combine contributions of different Feynman diagrams into gauge invariant structures, opening up the road to performing a number of calculations beyond the reach of standard methods, such as five-gluon-amplitudes [3] and four-graviton-amplitudes [4]. Strassler [5] later succeeded in deriving essentially the same set of rules without explicit reference to string theory, using path-integrals on the circle, and this reformulation of the Bern-Kosower method appears to be particularly well-suited to the calculation of effective actions [6,7].

In this paper, we demonstrate the efficiency of the method by performing the – to our knowledge first – calculation to order $\mathrm{O}(T^7)$ of the higher derivative expansion of the effective action for an external scalar potential induced by a scalar loop (in this special case the method reduces in principle to earlier work by Polyakov [8]). In spite of the absence of gauge symmetry, the string method turns out to hold significant advantages over standard methods both regarding the calculation of coefficients and the redundancy of the resulting operator basis, a fact which we will try to elucidate by comparing with several other types of heat kernel calculations. The connection with Feynman diagram calculations will also be clarified.

This special case of the higher derivative expansion is physically interesting in cases where the gauge field degrees of freedom are (at least in principle) already integrated out, or where they can be eliminated in another way. The latter is the case in sphaleron calculations in the 1+1-dimensional Abelian Higgs model [9], where the one-loop fluctuation determinants are determinants of Pöschl-Teller-operators. They can be calculated exactly [10], making the model an interesting playing ground for studying the convergence of the higher derivative expansion, as a useful preparation for the case of the electroweak sphaleron where the higher derivative is essential as those nonapproximative results which are available do contradict each other [11]. The first case appears most prominently in the discussion of radiative corrections to the bounce configuration in the electroweak phase transition [12]. In this case an accuracy beyond the order $\mathrm{O}(T^6)$ is highly desirable in order to make the discussion more quantitative.

The application of the string method to the calculation of the higher derivative expansion of the effective action with both a scalar potential and a nonabelian gauge field will be treated in a separate publication [13].

The higher derivative expansion of an effective action exists in two versions, which differ by the grouping of terms. The one we will consider is commonly obtained by writing the one-loop determinant in the Schwinger proper time representation

$$\Gamma[V] = -\log(\det M) = -\mathrm{Tr}(\log M) = \int_0^\infty \frac{dT}{T} \mathrm{Tr}\, e^{-TM} \tag{1}$$

and expanding in powers of T. For the case we will be treating in this paper,

$$M = -\partial^2 + m^2 + V(x) \tag{2}$$

with $V(x)$ some matrix–valued function on spacetime, this expansion has been previously calculated to order $\mathrm{O}(T^6)$ by various methods [9,14].

Alternatively, some authors prefer to calculate the same series up to a fixed number of derivatives, but with an arbitrary number of fields or potentials [15–17].

Let us first present the calculation by the string–inspired method, and compare with other methods later. For the sake of completeness, we will start from scratch and recapitulate how formula (1) for the effective action may be transformed into a worldline path integral (see e.g. [18]).

All calculations will be performed in d-dimensional euclidean spacetime; operator traces will be denoted by Tr , and ordinary matrix traces by tr .

Starting with the proper time integral

$$\Gamma[V] = \int_0^\infty \frac{dT}{T} e^{-m^2 T}\, \mathrm{Tr}\, exp\big[-T(-\partial^2 + V)\big] \tag{3}$$

one evaluates the trace in x–space and decomposes the interval T in the exponential, giving ($x_N = x_0 = x$)

$$\begin{aligned}
\Gamma[V] &= \int_0^\infty \frac{dT}{T} e^{-m^2 T} \mathrm{tr} \int dx < x \mid exp\big[-T(-\partial^2 + V)\big] \mid x > \\
&= \int_0^\infty \frac{dT}{T} e^{-m^2 T} \mathrm{tr} \int \prod_{i=1}^{N} dx_i < x_i \mid exp\left[-\frac{T}{N}(-\partial^2 + V)\right] \mid x_{i-1} > \\
&= \int_0^\infty \frac{dT}{T} e^{-m^2 T} \mathrm{tr} \int \prod_{i=1}^{N} dx_i K_{i,i-1}
\end{aligned} \tag{4}$$

The kernel $K_{i,i-1}$ may be evaluated by insertion of a momentum basis $\{\mid p_i >\}$,

$$\begin{aligned}
K_{i,i-1} &= \int \frac{dp_i}{(2\pi)^d} < x_i \mid exp\left[-\frac{T}{N}(-\partial^2 + V)\right] \mid p_i >< p_i \mid x_{i-1} > \\
&= \int \frac{dp_i}{(2\pi)^d}\, exp\left[-\frac{T}{N}\left(p_i^2 + V(x_i) + ip_i \frac{(x_{i-1} - x_i)}{T/N}\right)\right]
\end{aligned} \tag{5}$$

Taking N to be large and performing the gaussian momentum integrations yields

$$K_{i,i-1} = \frac{1}{(4\pi T/N)^{\frac{d}{2}}}\, exp\left[-\frac{T}{N}\big(\frac{\dot{x}_i^2}{4} + V(x_i)\big)\right], \tag{6}$$

and in the limit of $N \to \infty$ one obtains a representation for Γ as a quantum mechanical path integral on the space of closed paths in spacetime:

$$\Gamma[V] = \int_0^\infty \frac{dT}{T} [4\pi T]^{-\frac{d}{2}} e^{-m^2 T} \mathrm{tr}\, \mathcal{P} \int_{x(T)=x(0)} \mathcal{D}x\, exp\Big[-\int_0^T d\tau \Big(\frac{\dot{x}^2}{4} + V(x)\Big)\Big], \tag{7}$$

where $\mathcal{P}$ denotes path ordering and the normalization of the path integral is such that

$$\mathrm{tr}\, \mathcal{P} \int \mathcal{D}x\; exp\Big[-\int_0^T d\tau \Big(\frac{\dot{x}^2}{4}\Big)\Big] = 1 \tag{8}$$

The existence of such path integral representations for one-loop amplitudes is, of course, nothing new in principle [19 – 21]. The novel feature of the string-inspired method consists rather in the mode of their evaluation, which is by Wick contractions on the worldline, using the one-dimensional Green function for the Laplacian on the circle [8]:

$$\langle x^\mu(\tau_1) x^\nu(\tau_2)\rangle = -g^{\mu\nu} G(\tau_1,\tau_2) = \quad -g^{\mu\nu}\left[\mid \tau_1 - \tau_2 \mid - \frac{(\tau_1-\tau_2)^2}{T}\right] \tag{9}$$

This Green function can, however, not be applied to the path integral as it stands. Partial integration on the circle yields the identity

$$\int_0^T d\tau_2\, G(\tau_1,\tau_2)\ddot{x}(\tau_2) = 2x(\tau_1) - \frac{2}{T}\int_0^T d\tau_2\, x(\tau_2) \qquad , \tag{10}$$

where the second term should vanish, indicating that one should really integrate over the relative coordinate only. Let us therefore introduce the loop center of mass x_0, writing

$$x^\mu(\tau) = x_0^\mu + y^\mu(\tau) \tag{11}$$

with

$$\int_0^T d\tau\, y^\mu(\tau) = 0, \tag{12}$$

and extract the integral over the center of mass from the path integral:

$$\int \mathcal{D}x = \int d^d x_0 \int \mathcal{D}y. \tag{13}$$

Then Taylor-expanding the interaction part with respect to the center of mass,

$$V(x(\tau)) = e^{y\partial} V(x_0),$$

and expanding the path-ordered interaction exponential we get

$$\Gamma[V] = \int_0^\infty \frac{dT}{T}[4\pi T]^{-\frac{d}{2}} e^{-m^2 T} \mathrm{tr} \int d^d x_0 \sum_{n=0}^\infty \frac{(-1)^n}{n} T \int_0^{\tau_1 = T} d\tau_2 \int_0^{\tau_2} d\tau_3 \ldots \int_0^{\tau_{n-1}} d\tau_n$$
$$\times \int \mathcal{D}y\, e^{y(\tau_1)\partial_{(1)}} V^{(1)}(x_0) \ldots e^{y(\tau_n)\partial_{(n)}} V^{(n)}(x_0)\, exp\Big[-\int_0^T d\tau\, \frac{\dot{y}^2}{4}\Big]. \tag{14}$$

We have labeled the background fields $V^{(1)}, \ldots, V^{(n)}$, and the first τ – integration has been eliminated by using the freedom of choosing the point 0 somewhere on the loop, which is also the origin of the factor of $\frac{1}{n}$.

The path integrals for fixed n can now be performed using the Wick contraction formula for exponentials well-known from string theory,

$$\langle e^{y(\tau_1)\partial_{(1)}} e^{y(\tau_2)\partial_{(2)}} \rangle = e^{-G(\tau_1,\tau_2)\partial_{(1)}\partial_{(2)}} \tag{15}$$

yielding

$$\int \mathcal{D}y\, e^{y(\tau_1)\partial_{(1)}} V^{(1)}(x_0) \ldots e^{y(\tau_n)\partial_{(n)}} V^{(n)}(x_0) exp\Big[-\int_0^T d\tau\, \frac{\dot{y}^2}{4}\Big] =$$
$$exp\Big[-\sum_{i<k} G_B(\tau_i,\tau_k)\partial_{(i)}\partial_{(k)}\Big] V^{(1)} \ldots V^{(n)}. \tag{16}$$

Finally, by a rescaling

$$\tau_i = T\, u_i, \qquad i = 1, \ldots, n \tag{17}$$

and using the scaling property of G

$$G(\tau_1, \tau_2) = T\, G(u_1, u_2) \tag{18}$$

we arrive at

$$\Gamma[V] = \int_0^\infty \frac{dT}{T}[4\pi T]^{-\frac{d}{2}} e^{-m^2 T} \mathrm{tr} \int d^d x_0 \sum_{n=0}^\infty \frac{(-T)^n}{n}$$
$$\times \int_0^{u_1 = 1} du_2 \int_0^{u_2} du_3 \ldots \int_0^{u_{n-1}} du_n\, exp\Big[-T \sum_{i<k} G_B(u_i,u_k)\partial_{(i)}\partial_{(k)}\Big] V^{(1)} \ldots V^{(n)}. \tag{19}$$

The higher derivative expansion to some fixed order in T can now be simply obtained by expanding the exponential, performing a number of multi-integrals with polynomial integrands, which can be easily done with MAPLE or MATHEMATICA, and identifying operators which differ only by the cyclic ordering under the trace.
To order $\mathrm{O}(T^7)$ this can be done without excessive hardship, and leads to the following result:

$$\Gamma[V] = \int_0^\infty \frac{dT}{T} [4\pi T]^{-\frac{d}{2}} e^{-m^2 T} \sum_{n=1}^{7} (-T)^n \mathrm{tr}\, O_n \,, \tag{20}$$

where, using some shorthand-notation ($\partial_{\kappa\lambda} V := \partial_\kappa \partial_\lambda V(x_0)$ etc.) and changing x_0 to x,

$$O_1 = \int dx \Big(V \Big)$$

$$O_2 = \frac{1}{2!} \int dx \Big(V^2 \Big)$$

$$O_3 = \frac{1}{3!} \int dx \Big(V^3 + \frac{1}{2} \partial_\kappa V \partial_\kappa V \Big)$$

$$O_4 = \frac{1}{4!} \int dx \Big(V^4 + 2V \partial_\kappa V \partial_\kappa V + \frac{1}{5} \partial_{\kappa\lambda} V \partial_{\kappa\lambda} V \Big)$$

$$\begin{aligned} O_5 = \frac{1}{5!} \int dx \Big(& V^5 + 3V^2 \partial_\kappa V \partial_\kappa V + 2V \partial_\kappa V V \partial_\kappa V + V \partial_{\kappa\lambda} V \partial_{\kappa\lambda} V + \frac{5}{3} \partial_\kappa V \partial_\lambda V \partial_{\kappa\lambda} V \\ & + \frac{1}{14} \partial_{\kappa\lambda\mu} V \partial_{\kappa\lambda\mu} V \Big) \end{aligned}$$

$$\begin{aligned} O_6 = \frac{1}{6!} \int dx \Big(& V^6 + 4V^3 \partial_\kappa V \partial_\kappa V + 6V^2 \partial_\kappa V V \partial_\kappa V + \frac{12}{7} V^2 \partial_{\kappa\lambda} V \partial_{\kappa\lambda} V \\ & + \frac{9}{7} V \partial_{\kappa\lambda} V V \partial_{\kappa\lambda} V + \frac{26}{7} V \partial_{\kappa\lambda} V \partial_\kappa V \partial_\lambda V + \frac{26}{7} V \partial_\kappa V \partial_\lambda V \partial_{\kappa\lambda} V \\ & + \frac{17}{14} \partial_\kappa V \partial_\lambda V \partial_\kappa V \partial_\lambda V + \frac{18}{7} V \partial_\kappa V \partial_{\kappa\lambda} V \partial_\lambda V + \frac{9}{7} \partial_\kappa V \partial_\kappa V \partial_\lambda V \partial_\lambda V \\ & + \frac{3}{7} V \partial_{\kappa\lambda\mu} V \partial_{\kappa\lambda\mu} V + \partial_\mu V \partial_{\kappa\lambda} V \partial_{\kappa\lambda\mu} V + \partial_\mu V \partial_{\kappa\lambda\mu} V \partial_{\kappa\lambda} V \\ & + \frac{11}{21} \partial_{\kappa\lambda} V \partial_{\lambda\mu} V \partial_{\mu\kappa} V + \frac{1}{42} \partial_{\kappa\lambda\mu\nu} V \partial_{\kappa\lambda\mu\nu} V \Big) \end{aligned}$$

$$
\begin{aligned}
O_7 = \frac{1}{7!}\int dx\Big(& V^7 + 5V^4\partial_\kappa V\partial_\kappa V + 8V^3\partial_\kappa V V\partial_\kappa V + \frac{9}{2}V^2\partial_\kappa V V^2\partial_\kappa V + \frac{5}{2}V^3\partial_{\kappa\lambda}V\partial_{\kappa\lambda}V \\
& + \frac{9}{2}V^2\partial_{\kappa\lambda}V V\partial_{\kappa\lambda}V + 6V^2\partial_{\kappa\lambda}V\partial_\kappa V\partial_\lambda V + 6V^2\partial_\kappa V\partial_\lambda V\partial_{\kappa\lambda}V + \frac{7}{2}V^2\partial_\kappa V\partial_{\kappa\lambda}V\partial_\lambda V \\
& + \frac{17}{2}V\partial_\kappa V\partial_\lambda V\partial_\kappa V\partial_\lambda V + \frac{7}{2}V\partial_\kappa V\partial_\kappa V\partial_\lambda V\partial_\lambda V + \frac{11}{2}V\partial_\kappa V\partial_\lambda V\partial_\lambda V\partial_\kappa V \\
& + \frac{11}{2}V\partial_\kappa V V\partial_{\kappa\lambda}V\partial_\lambda V + \frac{11}{2}V\partial_\kappa V V\partial_\lambda V\partial_{\kappa\lambda}V + \frac{17}{2}V\partial_{\kappa\lambda}V V\partial_\kappa V\partial_\lambda V \\
& + \frac{2}{3}V\partial_{\kappa\lambda\mu}V V\partial_{\kappa\lambda\mu}V + \frac{5}{6}V^2\partial_{\kappa\lambda\mu}V\partial_{\kappa\lambda\mu}V + \frac{5}{2}V\partial_{\kappa\lambda\mu}V\partial_{\kappa\lambda}V\partial_\mu V \\
& + \frac{17}{6}V\partial_{\kappa\lambda\mu}V\partial_\kappa V\partial_{\lambda\mu}V + \frac{5}{3}V\partial_{\kappa\lambda}V\partial_{\kappa\lambda\mu}V\partial_\mu V + \frac{17}{6}V\partial_{\kappa\lambda}V\partial_\mu V\partial_{\kappa\lambda\mu}V \\
& + \frac{5}{2}V\partial_\kappa V\partial_{\lambda\mu}V\partial_{\kappa\lambda\mu}V + \frac{5}{3}V\partial_\kappa V\partial_{\kappa\lambda\mu}V\partial_{\lambda\mu}V + \frac{5}{3}\partial_\kappa V\partial_\kappa V\partial_{\lambda\mu}V\partial_{\lambda\mu}V \\
& + \frac{11}{6}\partial_\kappa V\partial_{\lambda\mu}V\partial_\kappa V\partial_{\lambda\mu}V + \frac{35}{9}\partial_{\kappa\lambda\mu}V\partial_\kappa V\partial_\lambda V\partial_\mu V + \frac{11}{3}V\partial_{\kappa\lambda}V\partial_{\kappa\mu}V\partial_{\lambda\mu}V \\
& + \frac{35}{18}\partial_{\kappa\lambda}V\partial_\kappa V\partial_{\lambda\mu}V\partial_\mu V + \frac{35}{18}\partial_{\kappa\lambda}V\partial_\mu V\partial_{\lambda\mu}V\partial_\kappa V + \frac{97}{18}\partial_{\kappa\lambda}V\partial_{\kappa\mu}V\partial_\lambda V\partial_\mu V \\
& + \frac{43}{18}\partial_{\kappa\lambda}V\partial_{\kappa\mu}V\partial_\mu V\partial_\lambda V + \frac{1}{6}V\partial_{\kappa\lambda\mu\nu}V\partial_{\kappa\lambda\mu\nu}V + \frac{1}{2}\partial_\kappa V\partial_{\lambda\mu\nu}V\partial_{\kappa\lambda\mu\nu}V \\
& + \frac{1}{2}\partial_\kappa V\partial_{\kappa\lambda\mu\nu}V\partial_{\lambda\mu\nu}V + \frac{7}{10}\partial_{\kappa\lambda}V\partial_{\mu\nu}V\partial_{\kappa\lambda\mu\nu}V + \frac{16}{15}\partial_{\kappa\lambda}V\partial_{\kappa\mu\nu}V\partial_{\lambda\mu\nu}V \\
& + \frac{1}{132}\partial_{\kappa\lambda\mu\nu o}V\partial_{\kappa\lambda\mu\nu o}V\Big)
\end{aligned}
$$

We have not performed the final T–integration, as it simply yields a Γ–function at any fixed order of T, which will have poles for the part of the effective action which is going to be renormalized at one loop for the dimension of spacetime considered.

The following two points should be emphasized about this calculation:
i) The final result has been obtained without performing any partial integrations with respect to x.
ii) Cyclic invariance has not been broken, despite of the formal breaking of cyclicity implied by distinguishing the variable u_1.

The book-keeping has still been done by hand, though computerization is in progress, and should allow calculation to several more orders in the expansion.

In the commutative case our result may be written in a much simpler form as, with x_0 fixed, $V(x_0)$ may be extracted from the path integral in eq. (7):

$$\Gamma[V] = \int_0^\infty \frac{dT}{T}[4\pi T]^{-\frac{d}{2}} \int dx\, e^{-[m^2+V(x)]T} \sum_{n=0}^{7} (-T)^n \tilde{O}_n \,, \tag{21}$$

$$\tilde{O}_0 = 1$$

$$\tilde{O}_1 = 0$$

$$\tilde{O}_2 = 0$$

$$\tilde{O}_3 = \frac{1}{3!}\left(\frac{1}{2}\partial_\kappa V \partial_\kappa V\right)$$

$$\tilde{O}_4 = \frac{1}{4!}\left(\frac{1}{5}\partial_{\kappa\lambda} V \partial_{\kappa\lambda} V\right)$$

$$\tilde{O}_5 = \frac{1}{5!}\left(\frac{5}{3}\partial_\kappa V \partial_\lambda V \partial_{\kappa\lambda} V + \frac{1}{14}\partial_{\kappa\lambda\mu} V \partial_{\kappa\lambda\mu} V\right)$$

$$\begin{aligned}\tilde{O}_6 = \frac{1}{6!}\Bigg(&\frac{5}{2}(\partial_\kappa V \partial_\kappa V)^2 + 2\partial_\mu V \partial_{\kappa\lambda} V \partial_{\kappa\lambda\mu} V + \frac{11}{21}\partial_{\kappa\lambda} V \partial_{\lambda\mu} V \partial_{\mu\kappa} V \\ &+ \frac{1}{42}\partial_{\kappa\lambda\mu\nu} V \partial_{\kappa\lambda\mu\nu} V\Bigg)\end{aligned}$$

$$\begin{aligned}\tilde{O}_7 = \frac{1}{7!}\Bigg(&\frac{7}{2}\partial_\kappa V \partial_\kappa V \partial_{\lambda\mu} V \partial_{\lambda\mu} V + \frac{35}{9}\partial_{\kappa\lambda\mu} V \partial_\kappa V \partial_\lambda V \partial_\mu V \\ &+ \frac{35}{3}\partial_{\kappa\lambda} V \partial_\kappa V \partial_{\lambda\mu} V \partial_\mu V + \partial_\kappa V \partial_{\lambda\mu\nu} V \partial_{\kappa\lambda\mu\nu} V + \frac{7}{10}\partial_{\kappa\lambda} V \partial_{\mu\nu} V \partial_{\kappa\lambda\mu\nu} V \\ &+ \frac{16}{15}\partial_{\kappa\lambda} V \partial_{\kappa\mu\nu} V \partial_{\lambda\mu\nu} V + \frac{1}{132}\partial_{\kappa\lambda\mu\nu o} V \partial_{\kappa\lambda\mu\nu o} V\Bigg)\end{aligned}$$

Now let us compare with the heat-kernel calculations of [9,14], who have obtained the order $\mathrm{O}(T^6)$ of this expansion (see also [22] and references therein).

Belkov et al. [14] have been using the standard recursive x–space calculation of the heat-kernel coefficients for the operator M, based on the separation ansatz

$$< x \mid K(T) \mid y > \quad = \quad < x \mid K_0(T) \mid y > H(x, y; T) \tag{22}$$

where

$$\begin{aligned} K &= exp\big[-T(-\partial^2 + m^2 + V(x))\big] \\ K_0 &= exp\big[-T(-\partial^2 + m^2)\big]. \end{aligned} \tag{23}$$

Inserted into the heat-kernel equation

$$\frac{\partial}{\partial T}K + MK = 0 \tag{24}$$

with boundary condition

$$K(T=0) = 1\!\!1 \tag{25}$$

and using the known heat kernel of M_0 this yields a recursive relation for the coefficients of the expansion of H in powers of T,

$$H(x,y;T) = \sum_{k=0}^{\infty} a_k(x,y)\, T^k \tag{26}$$

This recursion process is too cumbersome to do without computer help, and not easy to program for $d > 1$ (on the other hand, for $d = 1$ programming is trivial and calculation easy up to $\mathrm{O}(T^{12})$). More seriously, the expressions for O_6 obtained by those authors are considerably more lengthy than our result for the same quantity, and we have not been able to find the necessary partial integrations to reduce them to our result.

Carson [9] (see also [23]) uses a nonrecursive variant of the heat-kernel method first proposed by Nepomechie [24], using an insertion of plane waves into the functional trace:

$$\mathrm{Tr}\, e^{-TM} = \mathrm{tr} \int dx \int \frac{dp}{(2\pi)^d} e^{-ipx} e^{-TM} e^{ipx} = e^{-Tm^2}\, \mathrm{tr} \int dx \int \frac{dp}{(2\pi)^d} e^{-Tp^2} e^{-T\tilde{M}} 1\!\!1 \tag{27}$$

with

$$\tilde{M} = -\partial^2 + V(x) - 2ip\partial \tag{28}$$

Now one expands $e^{-T\tilde{M}} 1\!\!1$ in T, and performs the momentum integration, usually after a rescaling of the momenta by $\frac{1}{\sqrt{T}}$ to avoid mixing of T– orders by the integration. As the final step, all derivatives have to be commuted to the right till they vanish by acting on the $1\!\!1$.

The main drawback of this procedure, at least for the case at hand, is to be seen in the creation of an enormous number of terms at intermediate stages of the calculations, which cancel in the end. Up to order $\mathrm{O}(T^5)$ we have verified that this calculation leads to an effective action which is related to our result by partial integrations. At order $\mathrm{O}(T^6)$ the result given by Carson is similarly compact as ours, however this is after a considerable number of partial integrations have been performed, as we have checked by redoing this calculation. For better comparison, we have pursued the calculation to order $\mathrm{O}(T^7)$ by this method, too, using FORM [25]. We needed 2 hours of CPU–time on a NEXT work station to arrive at an $\mathrm{O}(T^7)$-part which contains 155 terms before identification of cyclically permuted operators, compared with 143 arising with the string method. Those

143 terms, however, may be immediately reduced to the 37 of our final result, due to the fact that cyclic invariance has not been broken. This is not true for the heat-kernel calculation, where this step would already be extremely tedious to do by hand, and difficult to implement with FORM.
For the special case of the one-dimensional Pöschl-Teller potential

$$V_s(x) = -\frac{s(s+1)}{cosh^2 x}, \tag{29}$$

relevant for the 1+1–dimensional Abelian Higgs model, we have checked that the effective actions calculated to $\mathrm{O}(T^7)$ by all three methods do agree.

Much closer related to our calculation is the earlier one by Zuk [15], who obtained those terms in the $\mathrm{O}(T^6)$ expansion which do not contain more than four derivatives. This author used Onofri's graphical method [26] for the heat kernel expansion to arrive at the following expression for the k-derivative contribution to the one-loop effective lagrangian (for the case of a massless complex scalar):

$$\begin{aligned}
\mathcal{L}^1_k &= -\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{d}{2}} \mathrm{tr}\, f^{(k)}(x;T), \\
f^{(k)}(x;T) &= \sum_{n=1}^{k} f^{(k)}_n(x;T), \\
f^{(k)}_n(x;T) &= 2^{\frac{k}{2}}(-1)^n \sum_{\alpha\in\mathcal{G}(n,k)} w_\alpha \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{n-1}} d\tau_n\, Q_\alpha(\tau;T) \\
&\quad \times \Big[\prod_{i=1}^{n} \frac{1}{r_i^\alpha!} e^{-(\tau_{i-1}-\tau_i)V} \partial_{\mu\kappa\lambda\ldots} V^{(i)}\Big] e^{-\tau_n V}
\end{aligned} \tag{30}$$

Here $\mathcal{G}(n,k)$ is the set of ordered graphs corresponding to all ways of connecting n points on a line by links in such a way that no point remains unlinked, admitting self-links. To every link one associates a factor of

$$Q(\tau_i,\tau_j) \equiv \tau_j(1-\frac{\tau_i}{T}) \tag{31}$$

and a contracted pair of derivatives acting on $V^{(i)}, V^{(j)}$. Q_α is the product of all Q_{ij} for a graph, and $w_\alpha, \frac{1}{r_i^\alpha!}$ are combinatorial weight factors .

This is obviously very similar to our eq. (19). And indeed, we may obtain eq. (30) from eq. (19) by observing that

$$G_B(u_i,u_j) = Q(u_i,u_i) + Q(u_j,u_j) - 2Q(u_i,u_j) \qquad\qquad (u_i \geq u_j), \tag{32}$$

and using the relation

$$\sum_{i=1}^{n} \partial_{(i)} = 0 \tag{33}$$

(valid up to total derivatives when acting on $V^{(1)} \dots V^{(n)}$) to rewrite

$$\sum_{i<k} G_B(u_i, u_k) \partial_{(i)} \partial_{(k)} = -\sum_{i=1}^{n} Q(u_i, u_i) \partial_{(i)}^2 - 2 \sum_{i<k} Q(u_i, u_j) \partial_{(i)} \partial_{(j)} \tag{34}$$

However, this transformation creates a huge redundancy; for instance, our single term $\frac{1}{14} \partial_{\kappa\lambda\mu} V \partial_{\kappa\lambda\mu} V$ from O_5 splits up into eight equivalent terms under it.

Finally, we comment on the calculation of the higher derivative expansion by Feynman diagrams [27], restricting to the non-matrix case for simplicity. Let us thus look at a one-component scalar field theory with self-interaction $U(\phi)$,

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - U(\phi) \tag{35}$$

The euclidean one-loop effective action may be expressed as (see e.g. [28])

$$\Gamma(\phi) = -\mathrm{Tr} \log \left[-\partial^2 + m^2 + U''(\phi) \right] \tag{36}$$

and thus corresponds to the special case

$$U'' \circ \phi = V \tag{37}$$

in our previous notation. To calculate the higher derivative expansion up to order $\mathrm{O}(T^n)$, one would have to calculate the one-loop diagrams with up to n insertions of the vertex $U''(\phi)$, and Taylor-expand the diagram with k vertices to order $2(n-k)$ in the external momenta. Using Feynman-parameters, this leads to formulas similar to equation (19); for instance, in the case of the four-point diagram (choosing $U(\phi) = \frac{\lambda}{4!} \phi^4$ for definiteness)

after performing the momentum integrations one obtains an α–parameter representation

$$\Gamma_4[V] \sim \int_0^\infty \frac{dT}{T} [4\pi T]^{-\frac{d}{2}} e^{-m^2 T} \prod_{i=1}^{4} d\alpha_1 \dots d\alpha_4 \, \delta(T - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4) \, exp\big[-Q(p_i, \alpha_i)\big], \tag{38}$$

where

$$TQ = \alpha_2\alpha_4(p_1+p_2)^2 + \alpha_1\alpha_3(p_2+p_3)^2 + \alpha_1\alpha_4 p_1^2 + \alpha_1\alpha_2 p_2^2 + \alpha_2\alpha_3 p_3^2 + \alpha_3\alpha_4 p_4^2 \tag{39}$$

Using the transformation of variables

$$\begin{aligned} \alpha_1 &= T(u_1 - u_2) \\ \alpha_2 &= T(u_2 - u_3) \\ \alpha_3 &= T(u_3 - u_4) \\ \alpha_4 &= T - (\alpha_1 + \alpha_2 + \alpha_3) = T\big[1 - (u_1 - u_4)\big] \end{aligned} \tag{40}$$

and momentum conservation

$$p_1 + p_2 + p_3 + p_4 = 0 \tag{41}$$

the exponent may be easily transformed into the Koba-Nielsen form

$$T \sum_{i<j} p_i p_j \, G_{ij} = T\Big\{ p_1 p_2 \left[(u_1 - u_2) - (u_1 - u_2)^2\right] + p_1 p_3 \left[(u_1 - u_3) - (u_1 - u_3)^2\right] + \dots \Big\} \tag{42}$$

(this type of transformation has been dubbed the "stringy rearrangement" by Lam [29]). However, we believe that this rearrangement, which restores the symmetry among the external legs, is quite essential for obtaining a relatively compact result for the effective action at high orders.

Finally, we would like to caution that this compactness of our form for the effective action might be deceptive with regard to applications requiring the use of the equations of motion, as it uses the mixed derivatives only, and one would have to perform a large number of partial integrations to create box operators for all external legs. On the other hand, it is easy to see that only up to order $\mathrm{O}(T^4)$ the effective action may be written in terms of box operators alone; beginning with $\mathrm{O}(T^5)$ the introduction of mixed derivatives is unavoidable.

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